the term in $\dot{x}$ in Eq. (9) divided by the frequency, that is,

$$\Gamma \simeq A\eta + B\sqrt{(\eta\rho_n\omega)} \quad (12)$$

or, again,

$$\Gamma/\eta \sim A + B\sqrt{\rho\omega/\eta}, \quad (13)$$

where we have neglected the small term γ due to the clamping and internal friction of the material.

In Fig. 7 we have plotted Γ/η against $\sqrt{\rho\omega/\eta}$: The linear dependence, though not perfect, is convincing. The peculiar shape of the graph is due to the steep fall of η as T is decreased. The arguments about boiling apply here also. In this case, the second term dominates: Again, this is evidence for high-frequency, non-Stokes damping.

This is to be expected: The relevant lengths in the reed are larger than the viscous penetration depth, which is smaller than 2 μm even at $T = 1.4$ K. The smallest of the dimensions of the reed, its thickness, is 15 μm. The smallness of δ is due to the relatively high frequencies we use. Use of a longer, or more flexible, reed would allow us to investigate the Stokes regime and probably, by turning on the higher modes, to investigate the crossover to the non-Stokes behavior that is predominant in our experiment.

V. CONCLUSIONS

The vibrating reed provides a novel way of measuring the superfluid transition in helium, which is both accurate and relatively simple. At the same time, the experimental results show that a fairly simple analysis of the fluid mechanics involved is sufficient to account for the changes in frequency and damping of the reed when it is submerged in liquid helium.

ACKNOWLEDGMENTS

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Conduction current and the magnetic field in a circular capacitor

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From the perspective of Ampere's circuital law, either displacement current or conduction current can be viewed as the source for the magnetic field inside a circular capacitor that is slowly being charged. The Biot–Savart law is more selective. How it can be used with conduction current alone is shown. Also considered is the "leaky capacitor. Here it is shown that an isolated charged capacitor which discharges slowly in a homogeneous Ohmic dielectric produces no magnetic field anywhere. Alternatively, a field is produced if the conducting material is confined to a limited region. This field is calculated for a circular capacitor when only the material in the gap is conducting.

I. INTRODUCTION

Thirty years ago French, King, and Tessman began an experiment to measure the magnetic field caused by Maxwell's displacement current. They planned to measure the field between the plates of a capacitor as it was being slowly charged. They stopped their measurement when they realized that the displacement current is superfluous; the mag-

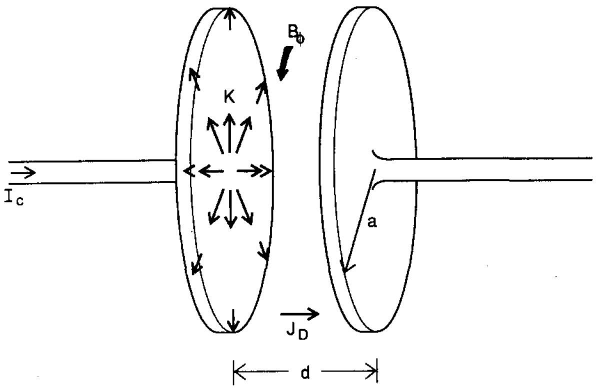

Fig. 1. Conduction currents and B field in a circular capacitor.

netic field in any quasistatic process can be calculated from the Biot–Savart law applied to conduction currents alone.[1,2]

More recently Corle and I have used a superconducting quantum interference device (SQUID) to make just such a measurement. We charged a circular parallel plate capacitor and verified that the magnetic field $B_\phi(\rho)$ inside the capacitor is as expected.[3] We learned of French and Tessman's work while our own was in progress. We were intrigued by their general theorem and have now developed a specific application to our capacitor.

Imagine a circular, parallel plate capacitor having plate of radius a and separation d. Our goal is to predict the magnetic field between the plates B_ϕ as caused by conduction currents alone. These are the radial currents K in the plates and the axial charging current I in the feed wires (see Fig. 1).

II. CIRCUITAL LAW

Ultimately we shall use the Biot–Savart law to predict the field. It is instructive, however, to begin using Ampere's circuital law. Idealize the capacitor by assuming that it is thin ($d \ll a$). Consider the loop shown in Fig. 2. In most texts,[4] Ampere's circuital law relates the integral of **B** around the loop to the flux of displacement current through the plane shaded area:

$$\left(\frac{c}{4\pi}\right)\oint \mathbf{B}\cdot d\mathbf{l} = \int \mathbf{J}_\mathbf{D}\cdot d\mathbf{A} = \left(\frac{1}{4\pi}\right)\int \frac{\partial D}{\partial t\, dA}$$
$$= \int\left(\frac{d\sigma}{dt}\right)dA = \frac{I\rho^2}{a^2}. \qquad (1)$$

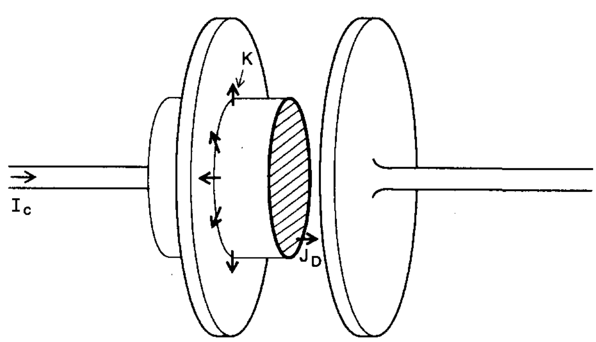

Fig. 2. B field from Ampere's circuital law. Shaded area penetrated by displacement current; unshaded area, by conduction current.

From this we find

$$B_\phi = 2I\rho/ca^2. \qquad (2)$$

Following Dahm, however, we may close the loop by another surface through which only current penetrates.[5] This is the unshaded surface in Fig. 2. The concealed end of this cylinder simply contributes the current I. Since the plates are being charged uniformly, the contribution of the curved part of the cylinder is just

$$2\pi\rho K = -(1-\rho^2/a^2)I. \qquad (3)$$

Thus the net effect of the current through the unshaded surface is

$$(c/4\pi)(2\pi\rho B_\phi) = I\rho^2/a^2, \qquad (4)$$

as found earlier in Eq. (2).

III. BIOT–SAVART LAW

The construction of Fig. 2 suggests a direct, if approximate, way of using the Biot–Savart law. For an observer between the plates, but far from the axis, the magnetic field from the feed wires is just that of an infinite wire,

$$B_\phi(\text{feed}) = 2I/c\rho. \qquad (5)$$

To this we must add the field owing to the radial conduction currents in the plates,

$$B_\phi(\text{plates}) = \frac{4\pi K}{c} = -\left(\frac{2I}{c\rho}\right)\left(1-\frac{\rho^2}{a^2}\right), \qquad (6)$$

where K is the radial surface conduction current density in the plates and the factor $(1-\rho^2/a^2)$ expresses the reduction in K arising from the uniform charging of the plates. Thus the total field is

$$B_\phi = 2I\rho/ca^2, \qquad (7)$$

for $\rho < a$ in agreement with the standard result found from using the displacement current and Ampere's circuital law.

Even for a thin capacitor our simple Biot–Savart treatment fails both close to the edge and close to the axis. Unfortunately, we cannot predict the magnetic fields close to the edge. This should not be surprising since near an edge one must use numerical techniques even for the electric field.[6]

We can, however, refine this calculation to predict the magnetic field close to the axis. This is an awkward region where the axial current in the feed wire is diverging to form the radial currents in the plates. The trick is to break the conduction current into two oppositely directed parts. (See Fig. 3.) The "continuous" current I_1 flows along the axis and thence radially within the plate to its edge. There it jumps to the second plate and flows back into the axis. This current produces neither charge on the plates nor magnetic field between them.[7] Both σ and B_ϕ result from the counterflowing "charging" current I_2. This nonconserved current begins on the axis of the second plate. It flows radially along the inner surface toward the edge. The strength of the charging current increases with ρ,

$$K_2 = I\rho/2\pi a^2, \qquad (8)$$

until at the edge its strength is equal to that of the continuous current. From here it jumps back to the first plate and flows into the axis of that plate. In the region between the plates I_1 and I_2 cancel, as they must since there is no net conduction current there.

The charging current K_2 does produce a magnetic field

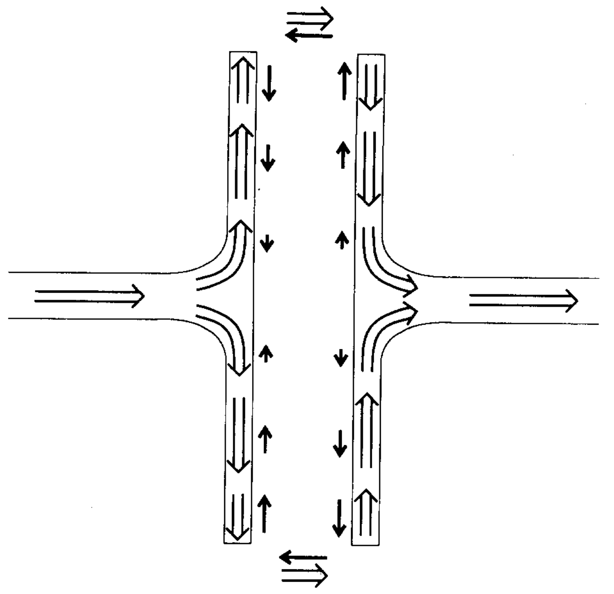

Fig. 3. B field close to the axis from the Biot–Savart law. Open arrows show "continuous" current I_1; closed arrows show "charging" current I_2.

at ρ. The contribution from the element $K_2(\rho',\phi',0)dA$ to the magnetic field $B_\phi(\rho,0,z)$ is simply

$$dB_\phi = K_2 z \cos\phi' \, dA / cR^3, \tag{9}$$

where $K_2 = I\rho'/2\pi a^2$, R is the distance from source to field point, and we have used a cylindrical coordinate system whose origin is the center of the given plate. The integral of dB_ϕ is most conveniently evaluated in a different cylindrical system: one whose axis passes through the field point. (See Fig. 4.) In this system K_2 has coordinates $(\rho'',\phi'',0)$ and the desired integral is

$$B_\rho = \left(\frac{I}{2\pi ca^2}\right)\int (\rho + \rho''\cos\phi'') \times z\rho'' \, d\phi'' \, d\rho'' (\rho''^2 + z^2)^{-3/2}. \tag{10}$$

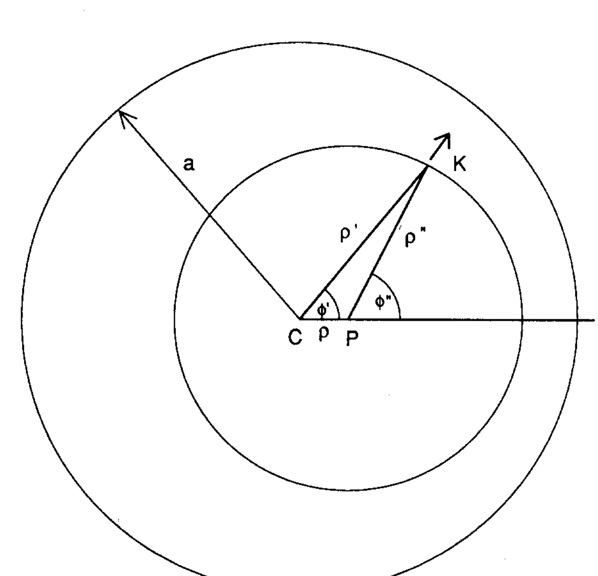

Fig. 4. Variables used in integration of Eq. (10): C = center of capacitor; P = field point.

Here we have used the relations $\rho'\cos\phi' = \rho + \rho''\cos\phi''$ and $dA = \rho'' \, d\rho'' \, d\phi''$.

Since we are mostly concerned with the field close to the axis ($\rho \ll a$), we consider first the near region, $\rho'' < a - \rho$. Here, ϕ'' can go through the full 2π rad without hitting the edge of the plate. Hence the integral of the $\cos\phi''$ term vanishes. The remaining integral is elementary,

$$B_\rho = Iz\rho(ca^2)^{-1}\int_0^{a-\rho} \rho'' \, d\rho'' (\rho''^2 + z^2)^{-3/2}$$
$$= I\rho(ca^2)^{-1}[1 - z/(a-\rho)]. \tag{11}$$

This is the dominant contributor to the integral. The two other sources are the lune-shaped region between the eccentric circle $\rho'' = a - \rho$ and the boundary of the plate $\rho' = a$ and I_2 flowing at $\rho' = a$ between the plates. Both these contribute only to terms of the order $I\rho(z/a)(a^2c)^{-1}$. In the limit of a thin capacitor with two plates, we are thus left with twice the leading term in Eq. (11),

$$B_\rho = 2I\rho/ca^2, \tag{12}$$

as desired.

We return now to the general question of why displacement currents are not needed for quasistatic magnetic fields. This is because the Biot–Savart law can be integrated by parts[8] to give

$$c\mathbf{B}(r) = \int \mathbf{J}(\mathbf{r}')\times\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d\tau' \tag{13}$$

$$= \int \nabla'\times\mathbf{J}\frac{d\tau'}{|\mathbf{r}-\mathbf{r}'|}. \tag{14}$$

Let us apply the second form of the Biot–Savart law, Eq. (14), to the displacement current $\mathbf{J}_D = (1/4\pi)\partial\mathbf{D}/\partial t$. We see immediately that the relation curl $\mathbf{E} = 0$ ensures that the quasistatic displacement current cannot produce a magnetic field either in vacuum or in a homogeneous dielectric. This fact was used by French and Tessman in their discussion of the leaky capacitor.

IV. LEAKY CAPACITOR

Consider a charged capacitor immersed in a conducting medium. Does the slow discharging of the capacitor produce a magnetic field? Naively we observe that the ionic current J_I between the plates is balanced by an equal and opposite displacement current. Thus there is no net current to produce a magnetic field. But French and Tessman have shown that when fringing fields are included the displacement current contributes nothing. We are then left with only the ionic current. Observe, however, that even this current is impotent if the lossy medium is homogeneous and ohmic. In that case $J_I \propto E \propto \nabla\phi$ everywhere and cannot contribute to Eq. (14). In the case of the leaky capacitor, naiveté might be justified after all.

We have seen that no magnetic field is produced if a capacitor of any geometry discharges in a homogeneous, ohmic medium. It is also evident that a nonconducting medium is permissible so long as it is not penetrated by any lines of force. Thus a thin capacitor of infinite radius will not produce a B field even if surrounded by nonconducting material.

Now consider a thin capacitor of finite radius a. Suppose that the capacitor is immersed in material of constant permeability, but that the material is conducting only in the

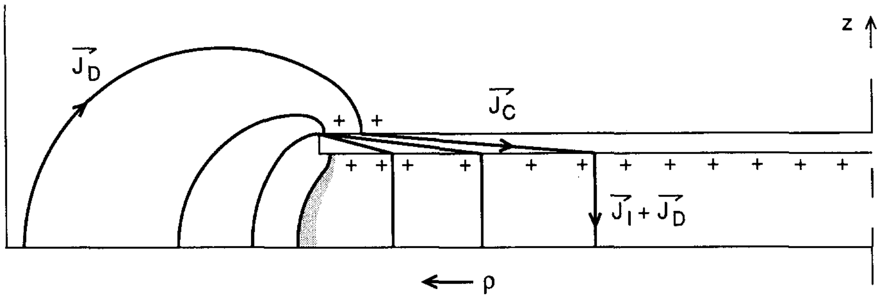

Fig. 5. Leaky capacitor discharges internally. Magnitude of conserved toroidal current is $I_D(\text{ext})$.

region between the plates. This is the case envisioned by French and Tessman.[2] Since the capacitor is no longer infinite, we might expect a B field about d/a times as large as that for a capacitor charged externally through axial leads.

Our expectation is correct. To see why, consider that at any time during the discharge the *total* displacement current I_D is equal and opposite to the ionic current I_I. Both currents may be divided into external (ext) and internal (int) parts according to whether the current is in the fringing or central field of the capacitor. For the displacement current, the ratio of these currents is just the ratio of the capacitances:

$$I_D(\text{ext}) = [C(\text{ext})/C(\text{tot})]I_D \equiv qI_D = -qI_I \qquad (15)$$

and

$$I_D(\text{int}) = [C(\text{int})/C(\text{tot})]I_D \equiv pI_D = -pI_I. \qquad (16)$$

For the leaky capacitor discharging internally, however, all the ionic current is internal,

$$I_I(\text{int}) = I_I; \quad I_I(\text{ext}) = 0. \qquad (17)$$

Both $I_I(\text{int})$ and $I_D(\text{int})$ are uniform. Thus we can use Ampere's circuital law to predict the B field within the plates $[r < (a-d)]$:

$$B_\phi = -2[I_I(\text{int}) - I_D(\text{int})]\rho/ca^2$$
$$= -2I_I(1-p)\rho/ca^2 = -2I_I\rho q/ca^2, \qquad (18)$$

where the minus sign arises because we have assumed the top plate to be positively charged, as in Fig. 5. This equation makes it clear that fringing fields are important. The fraction of capacitance q, represented by the external field, varies as d/a. Sloggett *et al.*[9] have recently determined that

$$\frac{q}{p} = \left(\frac{d}{\pi a}\right)\ln\left(\frac{16\pi a}{ed}\right) + \left(\frac{d}{2\pi a}\right)^2 \ln\left(\frac{16\pi a}{d}\right)^2. \qquad (19)$$

For the specific case of $d/a = \frac{1}{3}$ illustrated in Fig. 5 the ratio q/p is 0.49 and the external electric flux is fully half the internal.

Suppose that the location of the conducting material is reversed. It now occupies the space outside the plates rather than inside. A straightforward analysis, similar to the one just given shows that here $B_\phi = +2I_I p\rho/ca^2$. Table I summarizes the complementarity evidenced by internal and external conduction.

Internal discharge differs from homogeneous discharge in a key respect. The former requires conduction currents within the plates. Recall that the surface charge is uniformly distributed on the inside of a capacitor plate, but is strongly localized near the rim outside. When immersed in a homogeneous conducting medium both inside and outside charges are the direct source for the ionic current J_I. However, when the plate discharges internally (or externally) a radial conducting current J_C is needed within the plates to link the two charge distributions. Figure 5 illustrates an internally discharging capacitor.

Table I. Leaky capacitor. (All currents considered to be positive if flowing away from positively charged upper plate illustrated in Fig. 5.)

Condition	$I_D(\text{int})$	$I_D(\text{ext})$	$I_I(\text{int})$	$I_I(\text{ext})$	cB_θ
Complete immersion	$-pI_I$	$-qI_I$	pI_I	qI_I	0
Conducting material inside gap	$-pI_I$	$-qI_I$	I_I	0	$-2qI_I\rho/a^2$
Conducting material outside gap	$-pI_I$	$-qI_I$	0	I_I	$2pI_I\rho/a^2$

In the spirit of this paper, we should be able to understand the fields inside a leaky capacitor as arising from J_I and J_C alone. I will give the details separately. Here are the results for B_ϕ close to the axis of a thin capacitor that is leaking internally.

The contribution of the ionic current J_I may be calculated directly from the alternate form of the Biot–Savart law, Eq. (14). Here the volume integral reduces to a surface integral at the edge

$$cB_\phi = -J_I ad \int_0^{2\pi} \frac{d\phi \cos\phi}{|\mathbf{r} - \mathbf{r}'|}$$

$$= -J_I ad \int d\phi \cos\phi (a^2 - 2a\rho\cos\phi + \rho^2)^{-1/2}$$

$$\simeq -\pi J_I \rho d/a = -I_I \rho d/a^3. \quad (20)$$

The contribution of the ionic current is reinforced by a larger contribution from the conduction current I_C. For it we have

$$cB_\phi = 4\pi K_C = -2I_I q\rho/a^2 + I_I \rho d/a^3. \quad (21)$$

Here the first term originates from a ring source of current $I_C = I_D(\text{ext}) = qI_I$ at $\rho = a$ migrating to charge the inside of the plates uniformly. The second term is a correction term reflecting the fact that some of the charge on the *outside* of the plate is already present near the axis and thus does not have to flow radially.

In the limit $a \gg \rho, d$, the ratio of the contribution of the conduction and ionic currents to B_ϕ, grows logarithmically,

$$\frac{B_\phi(\text{cond})}{B_\phi(\text{ion})} \simeq \frac{2qa}{d} \simeq \left(\frac{2}{\pi}\right)\left(\frac{\ln 16\pi a}{ed}\right). \quad (22)$$

Evidently the purely ionic contribution will be hard to find.

V. CONCLUSIONS

Perhaps this discussion has only convinced the reader that it is fruitless to make delicate measurements of **B** inside a capacitor. As we have seen, this field may be predicted from the Biot–Savart law and conduction currents or from either conduction or displacement currents and the circuital law. Yet the very predictability may give interest to this relatively unexplored example of electromagnetism. For instance, in setting $\mathbf{J}_D = (1/4\pi) d\mathbf{D}/dt$, Maxwell assumed more than was needed. Strict current conservation could still be accommodated by adding to $\mathbf{J}_D$ the curl of **X** where **X** is an arbitrary vector field.[10] Such an anomalous **X** would produce a twisting of $\mathbf{J}_D$ which in turn produces a magnetic field in an unexpected direction. Recently Gengel and I have looked for such unexpected fields in a cylindrical capacitor.[11]

ACKNOWLEDGMENTS

I thank Timothy Corle and Anthony French for critical readings of this manuscript.

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HUXLEY—GARBAGE IN, GARBAGE OUT

Mathematics may be compared to a mill of exquisite workmanship, which grinds you stuff of any degree of fineness; but, nevertheless, what you get out depends upon what you put in; and as the grandest mill in the world will not extract wheat-flour from peascod, so pages of formulae will not get a definite result out of loose data.

Thomas Henry Huxley, "Geological Reform," Q. J. London **25**, xxxviii (1869). (Quoted in Joe D. Burchfield, *Lord Kelvin and the Age of the Earth* (Science History Publications, New York, 1975).